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Bounds on Codes Correcting Periodic Errors Blockwise

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ABSTRACT

This paper presents a necessary and sufficient condition on the number of parity check digits required for the existence of codes correcting *periodic errors* of different orders in different blocks of a code. An example of such a code has also been provided.

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1. INTRODUCTION

Investigations in coding theory have been made in several directions but one of the most important directions has been the detection and correction of errors. It began with Hamming [3] codes for single errors. Then Golay codes ([4], [5]) for double and triple random errors and thereafter BCH codes were studied for multiple error correction. There is a long history towards the growth of the subject and many of the codes developed have found applications in numerous areas of practical interest. One of the areas of practical importance in which growth of the subject took place is that of burst error detecting and correcting codes. It has also been observed that in many communication channels, burst errors occur more frequently than random errors. A burst of length b may be defined as follows:

Definition 1: *A burst of length b is a vector whose only non-zero components are among some b consecutive components, the first and the last of which is non-zero.*

It is clear that the nature of errors differ from channel to channel depending upon the behaviour of channels or the kind of errors which occur during the process of transmission. Many types of error patterns have been dealt with and codes have been constructed to combat such error patterns. Although the errors are classified mainly in two categories - random errors and burst errors, it has been observed that beyond these two categories also, errors may occur and follow certain patterns. These patterns are such that error repeats after some fixed interval. In certain communication channel like Astrophotography [9] where small mechanical error occurs periodically in the accuracy of the tracking in a motorized mount that results small movements of the target that can spoil long-exposure images, even if the mount is perfectly polar-aligned and appears to be tracking perfectly in short tests. It repeats at a regular interval - the interval being the amount of time it takes the mount's drive gear to complete one revolution. This type of errors is termed as *periodic or* *alternate errors*. Therefore, a code used for random or burst error correction is ineffective when used solely for the purpose of *periodic* error detection or correction, and so new codes must be developed. It was in this spirit that the codes detecting and correcting such errors were developed by Tyagi and Das ([1], [6]). A *periodic* error may be defined as follows:

Definition 2: *A periodic error of order s is an n- tuple whose non zero components are located at a gap of s positions and the number of its starting positions is among the first s+1 components, where s = 1,2,3,...,(n-1).*

For s =1, the *periodic* error of order 1 are the vectors where error may occur in 1st, 3rd, 5th,...... positions or 2nd, 4th, 6th,...... positions. For example, in a vector of length 8, *periodic* error vectors of order 1 are of the type 10101000, 00101000, 0010101, 10101010, 10001010, 01010101, 01000101, 00000101, 00000001 etc.

For s =2, the *periodic* error vectors of order 2 are those where error may occur in 1st, 4th, 7th,...... positions or 2nd, 5th, 8th,......positions or 3rd, 6th, 9th,.......positions. The *periodic* error vectors of order 2 may look like 10010010, 10000010, 00010010, 01001001, 01000001, 01000000, 00001001, etc in a vector of length 8.

For s =3, in a code length 8, the *periodic* error vectors of order 3 are 10001000, 01000100, 00100010, 00010001, 10000000, 01000000 etc.

The error correcting codes have been found to be powerful tools for checking the errors. The errors are usually detected and corrected in a block. When it is known that a particular type of error may occur within a specified number of digits, then if one desires to increase the block length, it is natural to expect some more errors among the additional digits. However, the errors, occurring in the additional digits, need not necessarily be of the type of errors as its earlier block. In view of this, the author has studied codes that can correct errors which are in the form of *periodic* errors having different orders in different blocks of a code word. The results have been derived in the case of two blocks. However, these can be extended to any finite number of blocks. Dass and Tyagi [2] studied linear codes that are capable of correcting block wise burst error. In this correspondence, this paper presents a parallel study in terms of *periodic* errors and obtains two results. This correspondence is organized as follows:

In section 2, the first result gives a necessary condition on the number of check digits required for the existence of a linear code over *GF(q)* that corrects all *periodic* errors of order s_1 in the first block of length n_1 and all *periodic* errors of order s_2 in the second block of length n_2. Section 3 gives the second result as sufficient condition on the number of check digits which ensures the existence of such a code. In section 4, an illustration of such a code is given along with a remark.

In what follows, the code length is taken to be n over *GF(q)*, consisting of two blocks of lengths n_1 and n_2 such that $n_1 + n_2 = n$. The distance between two vectors shall be considered in the Hamming sense.

2. A NECESSARY CONDITION

The following theorem gives a bound on the necessary number of parity check digits required for a code that corrects all *periodic* errors of order s_1 in the first block of length n_1 and all *periodic* errors of order s_2 in the second block of length n_2. The bound is based on the fact that the number of cosets is at least as large as the number of error patterns to be corrected (refer theorem 4.16 Peterson and Weldon [7]).

Theorem 1 *The number of parity check digits for an ($n_1 + n_2 = n$, k) linear code over GF(q) that corrects all periodic errors of order s_1 in the first block of length n_1 and all periodic errors of order s_2 in the second block of length n_2 is at least*

$$loq_q\left\{1+\sum_{i=0}^{s_1}(q^{k_i^1}-1)+\sum_{j=0}^{s_2}(q^{k_j^2}-1)\right\}$$

$$\text{where } k_i^1=\left\lceil\frac{n_1-i}{s_1+1}\right\rceil \text{ and } k_j^2=\left\lceil\frac{n_2-j}{s_2+1}\right\rceil.$$

Proof. This proof is based on counting the number of errors of above specific type and comparing with the available cosets in the ($n_1 + n_2 = n$, k) linear code over *GF(q)*.

Since the code is capable of correcting all errors which are all *periodic* errors of order s_1 in the first block of length n_1, all such *periodic* errors should be in different cosets; their number (refer theorem 1, Tyagi and Das[6]) is

$$\sum_{i=0}^{s_1}(q^{k_i^1}-1) \text{ where } k_i^1=\left\lceil\frac{n_1-i}{s_1+1}\right\rceil. \quad (1)$$

Similarly, as the code is capable of correcting all errors which are all *periodic* errors of order s_2 in the second block of length n_2, all such error patterns will be also in different cosets and their number is given by

$$\sum_{j=0}^{s_2}(q^{k_j^2}-1) \text{ where } k_j^2=\left\lceil\frac{n_2-j}{s_2+1}\right\rceil. \quad (2)$$

Thus the total number of errors including the zero-vector is

$$1+\sum_{i=0}^{s_1}(q^{k_i^1}-1)+\sum_{j=0}^{s_2}(q^{k_j^2}-1).$$

Since there must be at least this number of cosets and number of possible cosets is q^{n-k}, so

$$q^{n-k} \geq 1+\sum_{i=0}^{s_1}(q^{k_i^1}-1)+\sum_{j=0}^{s_2}(q^{k_j^2}-1), \quad (3)$$

$$\text{where } k_i^1=\left\lceil\frac{n_1-i}{s_1+1}\right\rceil \text{ and } k_j^2=\left\lceil\frac{n_2-j}{s_2+1}\right\rceil.$$ ■

3. A SUFFICIENT CONDITION

The paper now obtains an upper bound over the number of parity check digits sufficient for the existence of codes studied in **Theorem 1**. The proof is based on the technique used to establish Varshomov-Gilbert Sacks bound by constructing a parity check matrix for such a code (refer Sacks [8], also Theorem 4.17, Peterson and Weldon [7]). A method for constructing the parity check matrix for *periodic* error correcting codes in the case of only one block was given by Tyagi and Das [6].

Theorem 2 *Given positive integers s_1 and s_2, there exist an $(n_1 + n_2 = n, k)$ linear code over GF(q) that corrects all periodic errors of order s_1 in the first block of length n_1 and all periodic errors of order s_2 in the second block of length n_2 satisfying the inequality*

$$q^{n-k} > q^{p_2}\left\{\sum_{i=1}^{s_2}q^{k_i^2}-(s_2-1)+\sum_{i=0}^{s_1}(q^{k_i^1}-1)\right\}, \quad (4)$$

$$\text{where } p_2=\left\lceil\frac{n_2}{s_2+1}\right\rceil-1,\; k_i^1=\left\lceil\frac{n_1-i}{s_1+1}\right\rceil \text{ and } k_i^2=\left\lceil\frac{n_2-i}{s_2+1}\right\rceil.$$

Proof. The existence of such a code will be shown by constructing an appropriate $(n-k)\times n$ parity-check matrix H as follows:

After adding the first n_1 columns appropriately corresponding to the first block, the $(n_1+1)^{th}$, $(n_1+2)^{th}$, … columns are added to H so that the code corrects all *periodic* errors of order s_1 in the first block of length n_1 and all *periodic* errors of order s_2 in the second block of length n_2 . For this, the two requirements are needed to be satisfied:

(a) The syndrome resulting from the occurrence of any *periodic* error of order s_2 within the second sub block of length n_2 must be distinct from the syndrome resulting likewise from any other *periodic* error of order s_2 within *same* block.

(b) The syndrome resulting from the occurrence of any *periodic* error of order s_2 within the second block of length n_2 must be distinct from the syndrome resulting likewise from any *periodic* error of order s_1 within the *first* block of length n_1.

As the first requirement (a), the general $(n_1 + t)$ th column $(t > s_2+1)$ can be added to H such that it is not a linear combination of previous s_2-*periodic* columns $h_{n_1+t-(s_2+1)}$, $h_{n_1+t-2(s_2+1)}$, . . . , $h_{n_1+t-p_2(s_2+1)}$

where $p_2 = \left\lceil \frac{t}{s_2+1} \right\rceil - 1$, together with any other linear combination of s_2-*periodic* columns from the rest of the columns in the *same* block , i.e.,

$$h_{n_1+t} \neq \sum_{i=1}^{p_2} u_i h_{n_1+t-i(s_2+1)} + \sum_{i=0}^{k_r^2-1} v_i h_{n_1+t-r-i(s_2+1)} \; ; \quad r = 1, 2, \ldots, s_2 \tag{5}$$

where $u_i, v_i \in GF(q)$, $p_2 = \left\lceil \frac{t}{s_2+1} \right\rceil - 1$ and $k_r^2 = \left\lceil \frac{t-r}{s_2+1} \right\rceil$.

This constraint assures that the code which is the null space of the finally constructed matrix H will be capable of correcting all *periodic* errors of order s_2 in the second block of length n_2.

The number of possible linear combination of the R.H.S. of (5), including all zero vector, is given by (refer Tyagi and Das[6], theorem 2)

$$q^{p_2} \left\{ \sum_{i=1}^{s_2} q^{k_i^2} - (s_2 - 1) \right\} \tag{6}$$

where $p_2 = \left\lceil \frac{t}{s_2+1} \right\rceil - 1$ and $k_i^2 = \left\lceil \frac{t-i}{s_2+1} \right\rceil$.

By the second requirement (b), the $(n_1 + t)$ th column ($t > s_2+1$) cannot be a linear combination of previous s_2-*periodic* columns $h_{n_1+t-(s_2+1)}$, $h_{n_1+t-2(s_2+1)}$, . . . , $h_{n_1+t-p_2(s_2+1)}$ where $p_2 = \left\lceil \frac{t}{s_2+1} \right\rceil - 1$, together with any other linear combination of s_1-*periodic* columns among the columns $h_1, h_2, \ldots, h_{n_1}$, i.e.,

$$h_{n_1+t} \neq \sum_{i=1}^{p_2} u_i h_{n_1+t-i(s_2+1)} + \sum_{i=0}^{k_r^1-1} v_i h_{1+r+i(s_1+1)} \; ; \quad r = 1, 2, \ldots, s_1 \tag{7}$$

where $u_i, v_i \in GF(q)$ and $k_r^1 = \left\lceil \frac{n_1-r}{s_1+1} \right\rceil$.

This constraint assures that the code which is the null space of the finally constructed matrix H will also be capable of correcting all periodic errors of order s_1 in the first block of length n_1.

The number of ways in which the coefficients u_i 's can be selected is q^{p_2}.
And to enumerate the coefficients v_i 's is equivalent to enumerate the number of periodic errors of order s_1 in a vector of length n_1. This number of errors is given by (1).

Therefore all possible linear combination of R.H.S. of (7) is

$$q^{p_2} \times expr.(1) . \tag{8}$$

So the total number of columns to which h_{n_1+t}, cannot be equal is the sum of (6) and (8) i.e.,

$$q^{p_2} \left\{ \sum_{i=1}^{s_2} q^{k_i^2} - (s_2 - 1) \right\} + q^{p_2} \sum_{i=0}^{s_1} (q^{k_i^1} - 1)$$

i.e.,

$$q^{p_2} \left\{ \sum_{i=1}^{s_2} q^{k_i^2} - (s_2 - 1) + \sum_{i=0}^{s_1} (q^{k_i^1} - 1) \right\} \tag{9}$$

where $p_2 = \left\lceil \frac{t}{s_2+1} \right\rceil - 1$, $k_i^1 = \left\lceil \frac{n_1-i}{s_1+1} \right\rceil$ and $k_i^2 = \left\lceil \frac{t-i}{s_2+1} \right\rceil$.

At worst, all the linear combinations considered in (9) may be distinct.

Thus, while choosing the h_{n_1+t} th column, we must have

$$q^{n-k} > q^{p_2}\left\{\sum_{i=1}^{s_2} q^{k_i^2} - (s_2 - 1) + \sum_{i=0}^{s_1} (q^{k_i^1} - 1)\right\} \quad (10)$$

where $p_2 = \left\lceil \dfrac{t}{s_2+1} \right\rceil - 1$, $k_i^1 = \left\lceil \dfrac{n_1 - i}{s_1+1} \right\rceil$ and $k_i^2 = \left\lceil \dfrac{t-i}{s_2+1} \right\rceil$.

For completing the second block of length n_2, replacing t by n_2 gives the result as stated in (4). ■

The paper is concluded with an example and a remark.

Example : Consider a (6+4, 5) binary code with parity check matrix

$$H = \begin{bmatrix} 111000 & 0001 \\ 010111 & 0101 \\ 011010 & 1100 \\ 000011 & 0011 \\ 111111 & 1111 \end{bmatrix}_{5\times(6+4)}$$

This matrix has been constructed by the synthesis procedure outlined in the proof of Theorem 2, taking q=2, s_1=2, s_2=1, n_1=6 and n_2=4**.** The code which is the null space of the matrix given above corrects all *periodic* errors of order 2 in the first block of length 6 and all *periodic* errors of order 1 in the second block of length 4. It follows from the table 1 that all the error vectors and their corresponding syndromes which can be seen to be all distinct and non zero.

Table 1. Sub block error pattern

Error patterns	Syndromes	Error patterns	Syndromes
1st sub-block		2nd sub-block	
100000 0000	10001	000000 1000	00101
000100 0000	01001	000000 0010	00011
100100 0000	11000	000000 1010	00110
010000 0000	11101	000000 0100	01101
000010 0000	01111	000000 0001	11011
010010 0000	10010	000000 0101	10110
001000 0000	10101		
000001 0000	01011		
001001 0000	11110		

Remark It has been observed that deleting of the last row from the above parity check matrix gives rise to a (6+4, 6) binary code which satisfy the bound obtained in theorem 1 and it has been found the (6+4, 6) binary code is an **optimal** code in the sense that corrects all *periodic* errors of order 2 in the first block of length 6 and all *periodic* errors of order 1 in the second block of length 4 and *no other* error pattern.

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Pankaj Kumar Das was born in 1976 at Goalpara, Assam (India). He graduated from Goalpara College, Gauhati University, Assam in 1998. He completed his Master Degree in Mathematics from University of Delhi, Delhi in 2000. In 2005, he received his M.Phil. degree in Mathematics from the same university. He is working as an Assistant Professor in the Department of Mathematics, Shivaji College (University of Delhi), Delhi. He has given talks at national/international conferences. His area of research is Coding Theory.