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Entanglement classification – a comparative study of $SU(2)$ and $SL(2)$ developed operator model

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ABSTRACT

Entanglement classification is a core aspect of quantum information theory. It ensures successful quantum information processing. This article presents a comparative study of entanglement classification using developed operator models for the special unitary group and special linear group. This study was built upon prior work in entanglement classification in a pure three-qubit quantum system environment, where the operator models for each mathematical group were independently developed. Through extensive analysis, both synthesized models are functionally effective and yield the desired results. However, the comparative analysis reveals that the operator model exhibits certain limitations, particularly in its early phase of development compared to. This study provides significant enlightenments into the practical abilities of the developed operator models in entanglement classification and underlines the theoretical distinction between and paving the path for future research in quantum information theory, specifically entanglement classification.

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1. INTRODUCTION

An entanglement is regarded as the most important asset in quantum information theory, ensuring successfully executed information processing tasks [1], [2]. It is a strange yet fascinating phenomenon describing the relation of objects at a distance [3]-[5]. This particular asset has sparked massive interest regarding its vital role in various quantum technologies [6]-[17]. To harvest the full potential of the asset, a comprehensive understanding of its nature is vital.

An aspect of the understanding involves entanglement classification, which is a crucial process to classify the entanglement of quantum systems into their respective classes based on their entanglement properties [18]-[20]. Entanglement classification helps in determining the level or degree of entanglement. This guarantees effective information processing for quantum applications. Unfortunately, it is still considered a complex and poorly understood problem [21]-[23].

Various protocols have been designed specifically for this purpose, mainly local unitary (LU), local operations and classical communication (LOCC) and stochastic local operations and classical communication (SLOCC) [20]-[25]. These protocols offer valuable insights into the transformations of entangled systems. There are six inequivalent classes of entanglement under these protocols [18], [22]. One fully separable (A-B-C), three bi-separable (A-BC, B-AC, and C-AB) and two genuinely entangled (W and GHZ) classes. These classes represent the degree or level of entanglement that exists in those entangled quantum systems.

This research aims to investigate two developed operator models for the entanglement classification in pure three-qubit quantum systems, comprising special unitary groups and special linear groups. This study will provide valuable perspectives into the practicability of the developed models and the overall mathematical groups in entanglement classification. The outcome marks a milestone to the advancement of quantum computing technology, to a certain extent influence the communities of the world in achieving the sustainable development goals (SDG4, SDG8, SDG9, and SDG11) and possibly other SDGs in the near future [26]. This study is organized as follows. A step-by-step methodology of the development for both developed operator models is manifested in section 2. A detailed analysis and result of the operator models is described in section 3. Finally, section 4 concludes the study.

2. METHOD

This section describes the development process of $SU(2)$ and $SL(2)$ operator models using sets of specific generators coupled with their parameters resulting in the extended mathematical models.

2.1. Special unitary group operator model

The development process started with three sets of 2×2 matrices that were used as generators by means of dot product multiplication in the operator model development, producing a large 8×8 composite matrix representation originated from the mathematical model $SU(2) = e^{i\sigma_3\beta_1}e^{i\sigma_2\beta_2}e^{i\sigma_3\beta_3}$. Subsequently, through a parameter selection process, a set of selected parameters was implemented in the operator model. Figure 1 illustrates the modelling process of the operator model.

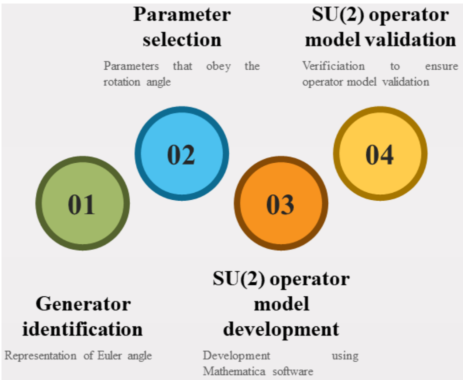

Figure 1. The $SU(2)$ operator model modeling process

Based on Figure 1, the development process started with understanding the parameterization of $SU(2)$. This phase determines the generator and parameters that will be used. In this development, a distinct, well-known, established representation in the field of particle physics was utilized. Then, the parameter values were determined and coordinated to the generator accordingly. Subsequently, enter the development process. The development and implementation of the 8×8 generated matrix was performed in the Mathematica 13.2 software, followed by the fully-developed of the $SU(2) \times SU(2) \times SU(2)$ operator model through cosine and sine functions, along with a series of exponential expansions.

In principle, the fully developed model is designed as an extended $SU(2)$ model, representing a three-qubit quantum system by means of integrating the operator model with an initial pure quantum state. The final phase is the operator model validation. This particular phase is vital to ensure that the developed operator model is accurate and reliable. Both $SU(2)$ and $SU(2) \times SU(2) \times SU(2)$ operator models were reviewed. The development of both developed models was compared with a manually calculated operator model, followed by principal comparisons with previous studies within the same mathematical model group. Figure 2 illustrates the overview process of the operator model development.

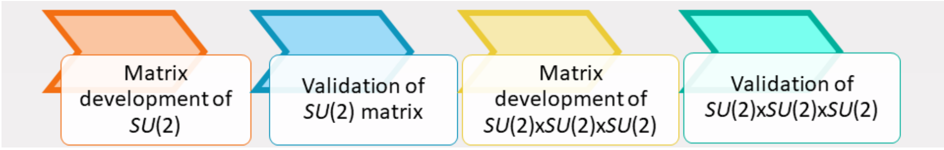

Figure 2. The operator model development process

2.2. Special linear group operator model

In principle, the development process is the same as the $SU(2)$ operator model. This process started with three sets of 2×2 matrices that were used as generators by means of dot product multiplication in the operator model development, producing a large 8×8 composite matrix representation originated from the mathematical model $SL(2) = e^{-itH}e^{wX}e^{rH}$. Thereafter, via a parameter selection process, a set of parameters selected was implemented in the operator model. Figure 3 depicts the modelling process of the operator model.

Based on Figure 3, the development process started with understanding the parameterization of $SL(2)$. This phase determines the generator and parameters that will be used. In this development, a distinct, well-known, established representation in the field of particle physics was utilized. Then, the parameter values were determined and coordinated to the generator correspondingly. Next, enter the development process. The development and implementation of the 8×8 generated matrix was executed in the Mathematica 13.2 software, followed by the fully-developed of the $SL(2) \times SL(2) \times SL(2)$ through cosine and sine functions, along with a series of exponential expansions.

In principle, the fully developed model is designed as an extended $SL(2)$ model, representing a three-qubit quantum system by means of integrating the operator model with an initial pure quantum state. The final phase is operator model validation. This phase is pertinent in ensuring that the developed operator model is accurate and reliable. Both $SL(2)$ and $SL(2) \times SL(2) \times SL(2)$ operator models were reviewed. The development of both was compared with a manually calculated operator model and then principally compared with previous studies within the same mathematical model group. Figure 4 illustrates the overview process of the operator model development.

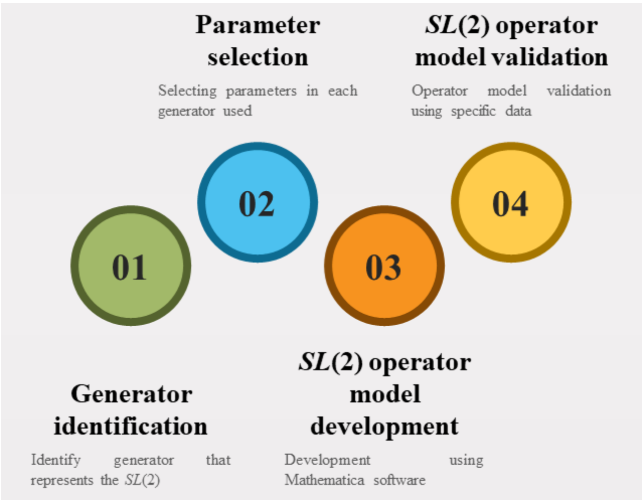

Figure 3. The $SL(2)$ operator model modeling process

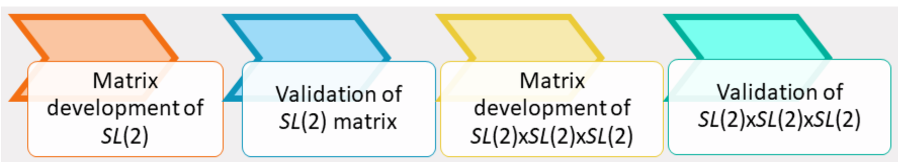

Figure 4. The operator model development process

3. RESULTS AND DISCUSSION

Both fully developed models are effectively functional and successfully produced the desired results. Adhering to the entanglement classification classes, the results are labelled accordingly. An exception particularly for bi-separable, “B-AC” and “C-AB” classes are not included in the analysis as the attributes for both classes are similar to “A-BC”. Additionally, the genuinely entangled classes are presented as “GE” as it portrays both W and GHZ classes. To further classify these classes, an extended measurement is required. Table 1 presents the entanglement classification results for both developed operator models.

Table 1. Entanglement classification results of $SU(2) \times SU(2) \times SU(2)$ and $SL(2) \times SL(2) \times SL(2)$ operator models

Special unitary group – $SU(2) \times SU(2) \times SU(2)$		Special linear group – $SL(2) \times SL(2) \times SL(2)$	
Initial quantum states	Final quantum states	Initial quantum states	Final quantum states
A-B-C	A-B-C	A-B-C	A-B-C
	A-B-C		A-B-C
	A-B-C		A-B-C
	A-B-C		A-B-C
A-BC	A-BC	A-BC	A-B-C
	A-BC		A-B-C
	A-BC		A-BC
	A-BC		A-BC
W	GE	W	A-B-C
	GE		A-B-C
	GE		GE
	GE		GE
GHZ	GE	GHZ	A-B-C
	GE		GE
	GE		GE
	GE		A-B-C

Fundamentally, both mathematical groups, the special unitary and special linear groups are unique and distinct from each other in terms of their characteristics and representations. Their fundamental differences highlight their respective roles in entanglement classification and quantum information processing in general. This enables experts to carefully make informed decisions on selecting the appropriate mathematical framework for specific quantum information processing tasks. Table 2 highlights the fundamental differences between special unitary and special linear groups.

Table 2. Fundamental differences between special unitary and special linear group

Special unitary group	Special linear group
Determined generator	Probabilistic generator
Suitable for pure state	Suitable for pure and mixed state
Preserve entanglement properties	May not preserve entanglement properties
Simple mathematical framework	Complex mathematical framework
Limited parameter space	Broad parameter space
Existence of practical challenges	Experimentally accessible with established techniques
Limited to certain quantum tasks	Applicable to various quantum information processing tasks

Table 2 reflects that both mathematical groups are different. The special unitary group employs a pre-determined generator and is well-suited for pure quantum states, where the entanglement properties of the quantum system are preserved. It offers a rather simpler mathematical framework but a limited parameter space. Its practical implementation may present certain challenges and the applicability is limited to certain quantum tasks.

On the contrary, the special linear group employs a probabilistic generator which makes it suitable for both pure and mixed quantum states. Even though it may not preserve the entanglement properties of the quantum system, its complex mathematical framework allows it to operate in a broader parameter space. In addition, it is experimentally accessible using established techniques, making it applicable to a wider range of quantum information processing tasks.

4. CONCLUSION

Conclusively, this comparative study of entanglement classification utilizing the two developed operator models for the special unitary group and special linear group in the context of a pure three-qubit quantum system environment has revealed clear distinctions between these two mathematical groups. As both models demonstrated functional effectiveness and achieved the desired results, the limitations of the developed $SU(2)$ operator model were also observed. The $SU(2)$ operator model with its deterministic generator, suitability for pure states, and entanglement-preserving properties, offers simplicity but exhibits constraints in terms of parameter space and potential practical challenges. In contrast, the $SL(2)$ operator model with its probabilistic generator, applicability to both pure and mixed states, and flexibility in parameter space, provides a more complex yet flexible approach that remains experimentally accessible. The theoretical and practical differences underscored in this study highlight the vital role of $SL(2)$ in accommodating a broader range of entanglement scenarios, making it a promising candidate for diverse quantum information processing tasks. This research enhances the understanding of entanglement classification and establishes a foundation for future investigations and works in quantum information theory, particularly on the limitations and potentials of both special unitary group and special linear group in specific quantum tasks, ultimately advancing the field of entanglement classification and quantum information theory.

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